S.-T. Yau College Student Mathematics Contests 2025

Oral Exams in Geometry and Topology

All-round (Solve 1 out of 2 problems)

1. Let S^n be the unit n-sphere and TS^n be its tangent bundle.

(1) Show that $TS^n \times \mathbb{R}$ is a trivial bundle over S^n.

(2) If $n = 2m$ is even, show that TS^{2m} is not a trivial bundle.

2. Let ω be a volume form on S^2. For each map $f : S^3 \to S^2$, the form $f^*\omega$ is a closed 2-form, so we can write $f^*\omega = d\alpha$ for some 1-form α. The number

$$H(f) = \int_{S^3} \alpha \wedge d\alpha$$

is called the *Hopf invariant* of f.

(1) Prove that $H(f)$ does not depend on the choice of α.

(2) Prove that homotopic maps have the same Hopf invariant.